

Greenberger-Horne-Zeilinger Paradoxes from Qudit Graph States

Weidong Tang^{1,2}, Sixia Yu^{1,2} and C.H. Oh²

¹*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics of University of Science and Technology of China, Hefei 230026, P.R. China*

²*Centre for Quantum Technologies and Physics Department, National University of Singapore, 2 Science Drive 3, Singapore 117542*

One fascinating way of revealing quantum nonlocality is the all-versus-nothing test due to Greenberger, Horne, and Zeilinger (GHZ) known as GHZ paradox. So far genuine multipartite and multilevel GHZ paradoxes are known to exist only in systems containing an odd number of particles. Here we shall construct GHZ paradoxes for an arbitrary number (greater than 3) of particles with the help of qudit graph states on a special kind of graphs, called GHZ graphs. Furthermore, based on the GHZ paradox arising from a GHZ graph, we derive a Bell inequality with two d -outcome observables for each observer, whose maximal violation attained by the corresponding graph state, and a Kochen-Specker inequality testing the quantum contextuality in a state-independent fashion.

Local realism cannot make quantum theory complete, as argued by Einstein, Podolsky, and Rosen (EPR) based on the belief that every element of physical reality must have a counterpart in a complete theory [1]. According to them, an element of reality is corresponding to a physical quantity whose value can be predicted with certainty without in any way disturbing a system. No disturbance is ensured by the locality, i.e., the assumption that the result of a measurement cannot be affected by any spacelike separated events. The clashing between the local realism and quantum mechanics as revealed by several no-go theorems such as Bell's theorem [2], Greenberger-Horne-Zeilinger (GHZ) theorem [3–5], and Kochen-Specker (KS) theorem [6], shows that the quantum mechanical description of our world is nonlocal, or more generally contextual. This fascinating and fundamental quantum feature of nonlocality and contextuality has been verified in experiments on various physical systems, e.g., [7], via the detection of violations of Bell inequalities and KS inequalities [8–10].

Among these genius approaches, GHZ theorem [3, 4] provides us an “all-versus-nothing” [11] test of a stronger type nonlocality, referred to as GHZ nonlocality, than Bell's nonlocality. This is a state-dependent argument: because of the perfect correlations in some special state called the GHZ state, e.g., a 3-qubit GHZ state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$, some local observables are elements of reality according to EPR. For example since the observable $\sigma_x^1 \sigma_y^2 \sigma_y^3$ stabilizes the GHZ state, i.e., $\sigma_x^1 \sigma_y^2 \sigma_y^3 |\Phi\rangle = |\Phi\rangle$, observables $\sigma_x^1, \sigma_y^2, \sigma_y^3$ are all elements of reality. Here $\sigma_{x,y,z}^k$ denote 3 standard Pauli matrices for the k th qubit. Similarly, from two other stabilizers $\sigma_y^1 \sigma_x^2 \sigma_y^3$ and $\sigma_y^1 \sigma_y^2 \sigma_x^3$ of $|\Phi\rangle$ we know that all $\sigma_{x,y}^k$ are elements of reality and must have realistic values $m_k^{x,y} = \pm 1$ for $k = 1, 2, 3$. Realistic values are supposed to obey the same algebraic relations as their corresponding observables. That is to say we have on the one hand $m_1^x m_2^x m_3^x = -1$, since $\sigma_x^1 \sigma_x^2 \sigma_x^3 |\Phi\rangle = -|\Phi\rangle$ and $m_1^x m_2^y m_3^y = m_1^y m_2^x m_3^y = m_1^y m_2^y m_3^x = 1$. On the other

hand, since $(m_k^y)^2 = 1$, we have identity $m_1^x m_2^x m_3^x = (m_1^x m_2^y m_3^y)(m_1^y m_2^x m_3^y)(m_1^y m_2^y m_3^x)$ which gives rise to a contradiction $-1 = 1$.

This elegant presentation of the GHZ paradox for 3 qubits is due to Mermin [5] soon after its first discovery for a 4-qubit GHZ state [3] and has already been verified experimentally [12]. Although originally the GHZ argument is state dependent, it was found recently that any GHZ paradox can give rise to a KS inequality for a state-independent test of quantum contextuality [8]. In addition to its fundamental role played in our understanding of quantum nonlocality and contextuality, the GHZ paradox also finds numerous applications such as in the quantum protocols for reducing communication complexity [13] and for secret sharing [14].

Compared to the bipartite and two-level case, multipartite and multilevel nonlocality or entanglement is poorly understood. In some quantum informational tasks such as quantum cryptography, the usage of multidimensional systems offers advantages such as an increased level of tolerance to noise at a given level of security and a higher flux of information compared to the two dimensional case [15]. Thus, it is crucial to investigate the relevant physical properties from some subclasses of these systems, e.g., GHZ nonlocality from a special kind of qudit states. Earlier efforts [16, 17] to generalize GHZ paradoxes to multidimensional and multilevel systems can be reduced either to the qubit cases or to fewer particle cases, except the cases of $n = 4j + 3$ for qubits [17]. Genuine multipartite multilevel GHZ paradoxes were first found by Cerf *et al.* for $(d + 1)$ -partite d -level systems with d being even [18]. An unconventional approach by using concurrent observables, not commuting yet having a common eigenstate, is proposed by Lee *et al.* to construct a GHZ paradox for the GHZ states of an odd number of particles [19]. Also a GHZ-like argument (all-versus-something) is proposed by Kaszlikowski *et al.* for d -partite d -level systems [20], in which concurrent observables have been used implicitly. Later, DiVincenzo and Peres [21] found out that not only can GHZ states exhibit

the GHZ paradox but also those code words, which are one kind of multipartite entangled states used in quantum error corrections [22], can exhibit GHZ nonlocality. But so far genuine multipartite and multilevel GHZ paradoxes for an even number of particles are still missing.

It turns out that GHZ states as well as code words from stabilizer codes [23] are graph states [24] which are essential resources for the one-way computing [25] and also provide an efficient construction of quantum error-correcting codes [26]. It is thus natural to take advantage of the perfect correlations in graph states for the constructions of GHZ paradoxes. In this Letter we shall identify those graphs, called GHZ graphs, whose corresponding graph states lead to genuine multipartite multilevel GHZ paradoxes. Furthermore we derive a Bell inequality for multipartite and multilevel systems as well as a state-independent KS inequality for every GHZ graph.

As a graph state for qubits is related to a simple graph, a nonbinary graph state [27–29] is associated with a weighted graph. Let $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$ denote the ring with addition modulo d . A \mathbb{Z}_d -weighted graph $G = (V, \Gamma)$ is composed of a set V of n vertices and a set of weighted edges specified by the adjacency matrix Γ , a symmetric $n \times n$ matrix with zero diagonal entries and the matrix element $\Gamma_{uv} \in \mathbb{Z}_d$ denoting the weight of the edge connecting the vertices u and v . A graph is *connected* if for any pair of vertices u, v there exists a finite number of vertices $\{v_i\}_{i=0}^K$ such that $\prod_{i=0}^{K-1} \Gamma_{v_i v_{i+1}} \neq 0$ with $u = v_0$ and $v = v_K$.

We denote by D_v the degree of vertex $v \in V$ which is the sum of the weights of all the edges connecting to v and by W the *total weight* of G which is the sum of the weights of all the edges. Explicitly, we have

$$D_v = \sum_{u \in V} \Gamma_{uv} \quad (v \in V), \quad W = \frac{1}{2} \sum_{u, v \in V} \Gamma_{uv}. \quad (1)$$

A GHZ graph is a connected \mathbb{Z}_d -weighted graph satisfying (i) the degree of each vertex is divisible by d , i.e., $D_v \equiv 0 \pmod{d}$, while (ii) the total weight is NOT divisible by d ; i.e., $W \not\equiv 0 \pmod{d}$. From these two conditions it follows immediately that the GHZ graph does not exist in odd dimensions and $\omega^W = -1$, where $\omega = e^{i\frac{2\pi}{d}}$. In fact, from the first condition, there is an integer t_v such that $D_v = dt_v$ for each $v \in V$, and from the fact that the total weight $W = dt/2$ with $t = \sum_{v \in V} t_v$ is an integer, since Γ is symmetric, it follows that if d is odd then t must be even and thus W is divisible by d . Furthermore, in even dimensions, the total weight W is not divisible by d if and only if t is odd and thus $\omega^W = (-1)^t = -1$. In what follows we shall always assume d to be even. A GHZ graph is called “primary” if for each vertex $a \in V$ there exists a pair of vertices b, c such that Γ_{ab} and Γ_{ac} are coprime and “weakly primary” if there exist three vertices $a, b, c \in V$, such that Γ_{ab} is coprime with Γ_{ac} .

In the case of $d = 2$ a GHZ graph has an odd number of edges and every vertex has an even number of neigh-

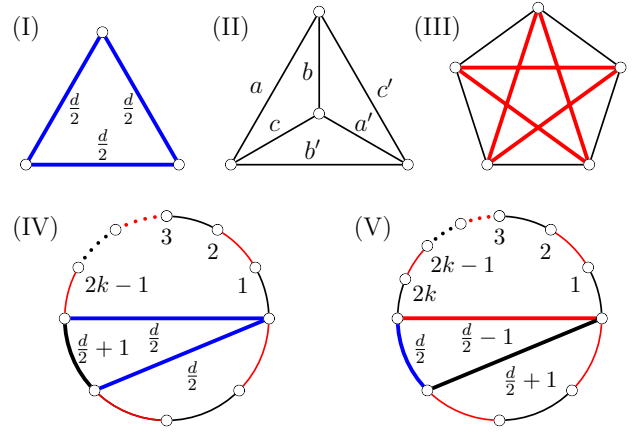


FIG. 1: Examples of GHZ graphs. Unlabeled thin black or red edges have weight 1 or $d-1$, respectively. All possible GHZ graphs on 3 and 4 vertices are shown in (I) and (II), where $a' = \frac{d}{2} + a$, $b' = \frac{d}{2} + b$, and $c' = \frac{d}{2} + c$ with $a+b+c = d/2$. A GHZ graph on 5 vertices is shown in (III) where the thick red edges have weight $d/2 - 1$. In (IV) and (V) two primary GHZ graphs on $2k+4$ and $2k+5$ vertices ($k \geq 1$) are shown.

bors. All GHZ graphs for $d = 2$ are primary. For example, a loop graph with an odd number of vertices and a complete graph with $4j+3$ ($j \geq 0$) vertices are possible GHZ graphs. There is only a single GHZ graph on 3 vertices as shown in Fig.1(I) and it is clear that it is not weakly primary if $d > 2$. In the case of $n = 4$ all possible GHZ graphs are shown in Fig.1(II) with weights satisfying $a+b+c = d/2$. If $d = 4k$ then $d/2 \pm 1 = 2k \pm 1$ are coprime and thus, by choosing, e.g., $a = 1, c = 1$, we obtain a primary GHZ graph. If $d = 4k+2$ then there always exists a vertex with all edges having even weights, since $d/2$ is odd, so that only a weakly primary GHZ graph exists in this case. Examples of primary GHZ graphs for arbitrary $n \geq 5$ and even dimensions are shown in Fig.1(III-V). The primary GHZ graph on 5 vertices as shown in Fig.1(III) can be generalized to any odd number of vertices.

Consider a system of n particles each of which has d energy levels, a *qudit* for short, and label them with V . Let $\{|s\rangle_v | s \in \mathbb{Z}_d\}$ be the computational basis for qudit $v \in V$ and $\{|\mathbf{s}\rangle | \mathbf{s} \in \mathbb{Z}_d^V\}$ is a basis for n qudits where \mathbb{Z}_d^V is the set of all n -dimensional vectors $\mathbf{s} = (s_1, s_2, \dots, s_n)$ with components $s_v \in \mathbb{Z}_d$ for all $v \in V$. To any weighted graph $G = (V, \Gamma)$ on $|V| = n$ vertices we can associate with a qudit graph state

$$|\Gamma\rangle = \frac{1}{d^{\frac{n}{2}}} \sum_{\mathbf{s} \in \mathbb{Z}_d^V} \omega^{\frac{1}{2} \mathbf{s} \cdot \Gamma \cdot \mathbf{s}} |\mathbf{s}\rangle, \quad (2)$$

which is also the unique joint $+1$ eigenstate of n commuting vertex stabilizers

$$g_v = X_v \prod_{u \in V} Z_u^{\Gamma_{uv}}, \quad (3)$$

i.e., $g_v|\Gamma\rangle = |\Gamma\rangle$ for all $v \in V$. Here we have introduced the generalized bit shift operator $X_v = \sum_{s \in \mathbb{Z}_d} |(s+1) \bmod d\rangle\langle s|_v$ and phase shift operators $Z_v = \sum_{s \in \mathbb{Z}_d} \omega^s |s\rangle\langle s|_v$ for each qudit $v \in V$. It is easy to check that $X_v^d = Z_v^d = I$ and $Z_v X_v = \omega X_v Z_v$. Our main result reads as follows:

Theorem For each (weakly) primary GHZ graph $G = (V, \Gamma)$ on $|V| = n$ vertices, with weights taken values in \mathbb{Z}_d , the graph state $|\Gamma\rangle$ provides a (weakly) genuine n -partite d -level GHZ paradox.

Before embarking on the proof we should clarify what we mean by *genuine n -partite and d -level* and give an example. According to [18] a GHZ paradox, formulated via a set of commuting observables, is said to be genuinely n -partite if one cannot reduce the number of parties and still have a Mermin-GHZ paradox. A GHZ paradox is (weakly) genuine d -level if one cannot reduce the dimensionality of the Hilbert space of (all) any one of the parties to less than d and still have a paradox.

As an example let us consider the GHZ graph as shown in Fig.1(II) in the case of $n = 4$ and the following 5 commuting observables that stabilize the corresponding graph state

$$\begin{array}{ccccc} X & Z^{a'} & Z^b & Z^c & +1 \\ Z^{a'} & X & Z^{c'} & Z^{b'} & +1 \\ Z^b & Z^{c'} & X & Z^a & +1 \\ Z^c & Z^{b'} & Z^a & X & +1 \\ X^\dagger & X^\dagger & X^\dagger & X^\dagger & -1 \end{array} \quad (4)$$

which provide us a GHZ paradox. Measurement of the product of the operators in each row gives a certainty result 1 or -1 as listed in the right column of Eq.(4) by quantum mechanics. With the analogue to EPR's argument, the result m_v^x or m_v^z of measuring the corresponding d -outcomes measurements X_v or Z_v on the v th qudit can be predicted in advance with certainty with the help of the results of spacelike separated measurements of X or Z on the other three qudits and are therefore elements of reality. Because the algebraic relations are preserved, we have $m_v^{x,z} = \omega^k$ with $\omega = e^{i2\pi/d}$ for some $k \in \mathbb{Z}_d$ and

$$\begin{aligned} (m_1^x)(m_2^z)^{a'}(m_3^z)^b(m_4^z)^c &= 1 \\ (m_1^z)^{a'}(m_2^x)(m_3^z)^{c'}(m_4^z)^{b'} &= 1 \\ (m_1^z)^b(m_2^z)^{c'}(m_3^x)(m_4^z)^a &= 1 \\ (m_1^z)^c(m_2^z)^{b'}(m_3^z)^a(m_4^x) &= 1 \\ (m_1^x)^{-1}(m_2^x)^{-1}(m_3^x)^{-1}(m_4^x)^{-1} &= -1. \end{aligned} \quad (5)$$

The contradiction lies in the fact that all five equations in Eq.(5) cannot hold simultaneously. In the case of $d = 4$ if we choose $a = b = 1$ and $c' = 2$ with $a' = b' = 3$ and $c = 0$ then the GHZ graph is primary and the corresponding GHZ paradox is genuine 4-partite and 4-level. In the case of $d = 6$ we can choose $a = b = c = 1$ and $a' = b' = c' = 4$ such that for the second qudit there exists a projection to

a qutrit by identification Z^2 for $d = 6$ with Z for $d = 3$. Thus it provides an example of weakly genuine 6-level GHZ paradox that can be regarded as GHZ paradox on a hybrid system of three 6-level system plus a qutrit.

Proof.— Let $G = (V, \Gamma)$ be a GHZ graph; i.e., the degree of each vertex D_a is divisible by d and the total weight W satisfies $\omega^W = -1$. For each qudit $v \in V$ we measure two unitary observables X_v and Z_v with outcomes assigned to values $m_v^x, m_v^z \in \{\omega^t | t \in \mathbb{Z}_d\}$, respectively. First of all these values are elements of reality because of the perfect correlations $g_v|\Gamma\rangle = |\Gamma\rangle$ ($v \in V$). In any local, or noncontextual, hidden variable models these values are independent of which observables might be measured by other observers. Furthermore, they must satisfy the same algebraic rules, e.g., the product rule, as their quantum counterparts do. For example from the definition of the vertex stabilizer g_v it follows

$$M_v := m_v^x \prod_{u \in V} (m_u^z)^{\Gamma_{uv}} = 1 \quad (6)$$

for each $v \in V$. On the other hand from the constraint $X_V|\Gamma\rangle = -|\Gamma\rangle$, because of the identity

$$\prod_{a \in V} g_a = \omega^W X_V \prod_{a \in V} Z_a^{D_a} = -X_V, \quad (7)$$

it follows that $\prod_{v \in V} m_v^x = -1$ which is impossible because $\prod_{v \in V} M_v = \prod_{v \in V} m_v^x$, in which the fact that D_v is divisible by d has been used.

By definition a GHZ graph is a connected graph and thus for each partition of n observers into two groups some of $n+1$ unitary observables will not be commuting when restricting to either one of two groups. Therefore the GHZ paradox for $|\Gamma\rangle$ is a genuine n partite. Furthermore, if the GHZ graph is primary then each vertex is attached to at least one pair of edges of coprime weights. If there were a projection to lower dimensions for a qudit, some eigenstates of X_a and those of $Z_a^{\Gamma_{ab}}$ and $Z_a^{\Gamma_{ac}}$ are orthogonal. This is impossible because first there always exist $p, q \in \mathbb{Z}_d$ such that $p\Gamma_{ab} + q\Gamma_{ac} = 1 \bmod d$ and, second, X_v and Z_v are two complementary observables whose eigenstates cannot have a zero overlap, which is the case if the dimensionality can be reduced. \sharp

Some remarks are in order. First, for we have constructed genuine n -partite and d -level GHZ paradox with $n \geq 5$ can be even. Second, any state that is related with GHZ graph states via local unitary transformations exhibits also GHZ nonlocality. Third, for a graph that is not GHZ graph it is also possible to construct a GHZ paradox for the graph state if the underlying graph contains a GHZ subgraph. A *subgraph* $H = (V', \Gamma')$ of a weighted graph $G = (V, \Gamma)$ is also a \mathbb{Z}_d -weighted graph with a vertex set given by $V' \subseteq V$ and edges specified by $\Gamma'_{ab} = \Gamma_{ab}$ if $a, b \in V'$. If furthermore the subgraph is a GHZ graph we shall refer to it as a *GHZ subgraph* of G . Suppose that the graph G contains a GHZ graph

$H = (V', \Gamma')$ with $|V'| = m < n$, then the $m + 1$ observables g_u with $u \in V'$ and $\prod_{u \in V'} g_u$ yield a GHZ paradox for the graph state $|\Gamma\rangle$. It is clear that it is only a genuine m -partite GHZ paradox if the GHZ subgraph is primary. For example, the 4-qubit GHZ state is equivalent to the graph state corresponding to the complete graph on 4 vertices, which contains a loop of length 3 as a GHZ subgraph. In fact the original GHZ proof [4] revealed a 3-partite GHZ nonlocality using this GHZ subgraph.

As the first application we shall derive a Bell inequality with two measurement settings for each observer with the help of the GHZ paradox derived from a GHZ graph. Consider two d -outcome measurements A_v and B_v for each observer $v \in V$ and assign values in $\{\omega^t | t \in \mathbb{Z}_d\}$ to them (Bell-KS value assignment). For each GHZ graph $G = (V, \Gamma)$ we introduce a Bell operator as

$$\mathcal{B}_G = \sum_{\substack{k=1 \\ k \text{ odd}}}^{d-1} \frac{2}{d} \left(\sum_{v \in V} A_v^k \prod_{u \in V} B_u^{k\Gamma_{uv}} - \prod_{v \in V} A_v^k \right) \quad (8)$$

Taking into account the identity $\sum_{k=0}^{d-1} \omega^{kl} = d\delta_{l,0}$ for arbitrary $l \in \mathbb{Z}_d$ and denoting $A_V = \prod_{v \in V} A_v$ and $N_v = A_v \prod_{u \in V} B_u^{\Gamma_{uv}}$ for each $v \in V$, where $\delta_{i,j}$ is the standard Kronecker delta symbol, we have

$$\mathcal{B}_G = \delta_{-1, A_V} - \delta_{1, A_V} + \sum_{v \in V} (\delta_{1, N_v} - \delta_{-1, N_v}) \leq n - 1. \quad (9)$$

The inequality holds in any local realistic theory because if there are n positive terms then there is necessarily a negative term in \mathcal{B}_G : If $N_v = 1$ for every $v \in V$ then it holds $A_V = 1$ which contributes a negative term; if $N_v = 1$ for all $v \in V - \{v_0\}$ and $A_V = -1$ then it necessarily holds $N_{v_0} = -1$ because $A_V = \prod_{v \in V} N_v$. Furthermore it is easy to see that $\mathcal{B}_G \leq n + 1$, which is attained by the graph state $|\Gamma\rangle$ with $\langle \Gamma | \mathcal{B}_G | \Gamma \rangle = n + 1$ in which A_v and B_v are chosen to be X_v and Z_v , respectively, for each $v \in V$. In this case the quantum to classical ratio $(n + 1)/(n - 1)$ is a constant independent of the dimension, comparing to that of [30].

Every GHZ paradox leads also to a proof of KS theorem. And any proof of KS theorem can be converted to an experimentally testable inequality, called as KS inequality, in the manner of Cabello [8]. As the second application we consider the following KS inequality

$$\frac{1}{2} \left\langle X_V^\dagger \prod_{v \in V} X_v + \sum_{v \in V} g_v^\dagger X_v \prod_{u \in V} Z_u^{\Gamma_{uv}} + H.c. \right\rangle_c - \frac{1}{2} \left\langle X_V^\dagger \prod_{v \in V} g_v + H.c. \right\rangle_c \leq C_{n+1,d} \quad (10)$$

where, with $\lambda = \frac{d}{2(n+1)}$ and $\theta = 2\pi/d$, we have denoted

$$\frac{C_{nd}}{n+1} = (\lambda - [\lambda]) \cos[\lambda\theta] + (1 + [\lambda] - \lambda) \cos[\lambda\theta]. \quad (11)$$

First, each term, e.g., $\left\langle X_V^\dagger \prod_{v \in V} g_v \right\rangle_c$, is the abbreviated form of the classical correlation of $n + 1$ observables, e.g., $\left\langle X_V^\dagger g_1 g_2 \dots g_n \right\rangle_c$. Second, the upper bound can be easily inferred from the Lemma proved below. Third, we have $C_{n+1,d} < n + 2$ while the quantum mechanical value of the left-hand side of Eq.(10) equals to $n + 2$ identically and therefore violates the above KS inequality in a state-independent fashion.

In summary, first of all we have identified a special kind of graphs, called GHZ graphs, whose corresponding graph states give rise to GHZ paradoxes. Except for the case $n = 4$ with $d = 4k + 2$ for which only a weakly genuine GHZ paradox is found we have derived genuine n -partite and d -level GHZ paradoxes from qudit graph states corresponding to GHZ graphs with $n \geq 4$ and even d being arbitrary. Second, as applications for each GHZ graph we derive a Bell inequality with two d -outcome observables for each observer whose maximal violation is attained by the corresponding graph state as well as a state-independent KS inequality that is satisfied by any noncontextual hidden variable models. This would be helpful to the analysis of multipartite contextuality or multipartite nonlocality. It should be noted that GHZ paradoxes may exist for those states that are equivalent to the graph states under local Clifford (LC) transformations. However the conditions under which both two GHZ paradoxes arising from two LC equivalent states are genuine n -partite seem to lie out of the reach of current Letter. Besides, the examples we are analyzing here involve only some special classes of graph states, so figuring out other classes of graph states which are consistent with our theorem are still meaningful as for fixed parties n , different graphs may have different robustness against decoherence, which may help to design new quantum protocols for reducing communication complexity. Ironically, a genuine 4-partite GHZ paradox is still missing for the original 4-qubit GHZ state.

This work is supported by National Research Foundation and Ministry of Education, Singapore (Grant No. WBS: R-710-000-008-271) and supported by the financial support of NNSF of China (Grant No. 11075227).

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- [1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
 - [2] J. S. Bell, Physics **1**, 195 (1964).
 - [3] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in *Bells Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69; arXiv: 0712.0921.
 - [4] D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
 - [5] N.D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
 - [6] S. Kochen and E.P. Specker, J. Math. Mech. **17**, 59 (1967).

- [7] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **47**, 460 (1981); R. Lapkiewicz *et al.*, Nature (London) **474**, 490 (2011); C. Zu, *et al.*, Phys. Rev. Lett. **109**, 150401 (2012).
- [8] A. Cabello, Phys. Rev. Lett. **101**, 210401 (2008).
- [9] P. Badziąg, I. Bengtsson, A. Cabello, and I. Pitowsky, Phys. Rev. Lett. **103**, 050401 (2009).
- [10] S. Yu and C.H. Oh, Phys. Rev. Lett. **108**, 030402 (2012).
- [11] N.D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
- [12] D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **82**, 1345 (1999); J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature(London) **403**, 515 (2000).
- [13] R. Cleve and H. Buhrman, Phys. Rev. A **56**, 1201 (1997).
- [14] M. Żukowski *et al.*, Acta Phys. Pol. A **93**, 187 (1998).
- [15] S. Gröblacher, T. Jennewein, A. Vaziri, G. Weihs, and A. Zeilinger, New J. Phys. **8**, 75 (2006).
- [16] A. Cabello, Phys. Rev. A **63**, 022104 (2001).
- [17] C. Pagonis, M.L.G. Redhead, and R.K. Clifton, Phys. Lett. A **155**, 441 (1991).
- [18] N.J. Cerf, S. Massar, and S. Pironio, Phys. Rev. Lett. **89**, 080402 (2002).
- [19] J. Lee, S.-W. Lee, and M.S. Kim, Phys. Rev. A **73**, 032316 (2006).
- [20] D. Kaszlikowski and M. Żukowski, Phys. Rev. A **66**, 042107 (2002).
- [21] D.P. DiVincenzo and A. Peres, Phys. Rev. A **55**, 4089 (1997).
- [22] P.W. Shor, Phys. Rev. A **52**, R2493 (1995).
- [23] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).
- [24] M. Hein, J. Eisert, and H.J. Briegel, Phys. Rev. A **69**, 062311 (2004).
- [25] R. Raussendorf and H.J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
- [26] S. Yu, Q. Chen, C.H. Lai, and C.H. Oh, Phys. Rev. Lett. **101**, 090501 (2008); S. Yu, Q. Chen, and C.H. Oh, arXiv: 0709.1780.
- [27] D. Hu, W. Tang, M. Zhao, Q. Chen, S. Yu, and C.H. Oh, Phys. Rev. A **78**, 012306 (2008).
- [28] D. Schlingemann and R.F. Werner, Phys. Rev. A, **65**, 012308 (2001).
- [29] D. Schlingemann, Quant. Inform. Comput. **2**, 307 (2002).
- [30] W. Son, J. Lee, and M.S. Kim, Phys. Rev. Lett. **96**, 060406 (2006).

Lemma Let $\mathbb{U}_d = \{\theta, 2\theta, \dots, d\theta\}$ with $\theta = 2\pi/d$ and $\lambda = \frac{d}{2(n+1)}$ and $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be n real variables and $x_s = \sum_{i=1}^n x_i$. We have

$$\max_{\vec{x} \in \mathbb{U}_d^n} \left\{ f(\vec{x}) := \sum_{i=1}^n \cos x_i - \cos(x_s) \right\} = C_{n,d}. \quad (12)$$

Specially, if $n \geq d/2$, i.e., $\lambda < 1$, then $C_{nd} = n + 1 - d \sin^2 \frac{\pi}{d}$ and if $d = 2(n+1)l$ for some l , i.e., λ is an integer, then $C_{nd} = (n+1) \cos \frac{\pi}{n+1}$.

Proof. The maximum of $f(\vec{x})$ over \mathbb{R}^n is the largest value on all local extremal points satisfying $\frac{\partial f(\vec{x})}{\partial x_i} = \sin x_s - \sin x_i = 0$ ($\forall i$). Let $x_1 = x$ then either $x_i = a_i = 2u_i\pi + x$ or $x_i = b_i = (2u_i + 1)\pi - x$ for all integers u_i with $i \geq 2$ since $\sin x_i = \sin x$. Denote by m the number of x_i 's being equal to a_i and $k = n - m$ the number of x_i 's being equal to b_i 's among $\{x_i\}_{i=1}^n$. Then from $\sin x = \sin(k\pi + (m-k)x)$ it follows either a) $x = x_a$ with $2l\pi + x_a = k\pi + (m-k)x_a$ or b) $x = x_b$ with $(2l+1)\pi - x_b = k\pi + (m-k)x_b$ for all the integers $l \geq 0$. At these extremal points we have either $f(\vec{x}_a) = (m-k-1) \cos x_a$ or $f(\vec{x}_b) = (m-k+1) \cos x_b$. If $k \geq 1$ then $m-k = n-2k \leq n-2$ and thus $f(\vec{x}_{a,b}) \leq n-1$. If $k = 0$ then $f(\vec{x}_l) = (n+1) \cos x_l$, where $\vec{x}_l = (x_l, x_l, \dots, x_l)$ with $x_l = (2l+1)\pi/(n+1)$. Since $f(\vec{x}_0) \geq n-1$ and $f(\vec{x}_0) \geq f(\vec{x}_l)$ the extremal point \vec{x}_0 leads to the largest value of f .

If λ is an integer then $\vec{x}_0 \in \mathbb{U}_d^n$ and thus the global maximum $f(\vec{x}_0)$ is attainable and in this case $f(\vec{x}_0) = C_{nd}$. If λ is not an integer then the maximal value $f(\vec{x}_0)$ is not attainable by any vector in \mathbb{U}_d^n . However its maximum must be attained at those vectors near one of those extremal points that have the floors or ceilings of the components of the extremal points. We consider at first those vectors in \mathbb{U}_d^n near \vec{x}_0 that have a number m of $x_+ = \lceil \lambda \rceil \theta$ and a number $n-m$ of $x_- = \lfloor \lambda \rfloor \theta$ as components with $n \geq m \geq 0$. On these vectors $f(\vec{x})$ assumes values $F_m = m \cos x_+ + (n-m) \cos x_- - \cos(m\theta + nx_-)$. Let $\Delta_m = (F_{m+1} - F_m)/(2 \sin \frac{\theta}{2})$ and we have

$$\Delta_m = \sin \frac{2nx_- + (2m+1)\theta}{2} - \sin \frac{x_- + x_+}{2} \quad (13)$$

Since λ is not an integer we have $0 \leq (x_- + x_+)/2 \leq \pi/2$. In the case of $d/2 > n$ we have $nx_- + m\theta \leq 2\pi + x_-$ and then $\Delta_m \geq 0$ if $m < \delta := d/2 - (n+1)\lfloor \lambda \rfloor$ and $\Delta_m \leq 0$ if $m > \delta$. As a result $\max F_m = F_\delta = C_{nd}$. If $n \geq d/2$ then $x_- = 0$ and in this case $\Delta_m \geq 0$ if $m'_u \leq m \leq m_u$ with $m'_u = ud$ or $m_u = (u+1/2)d - 1$ and $\Delta_m < 0$ otherwise for $u \geq 0$ being integer. Thus the maximum value of F_m must be taken on m'_u or m_u and obviously $F_{m_0} \geq F_{m_u}$ and $F_{m_0} \geq F_{m'_u}$ for all u . As a result we have $\max F_m = F_{m_0} = C_{nd}$. Since $C_{nd} \geq (n+1) \cos \lceil \lambda \rceil \theta \geq f(\vec{x}_l)$ for all $l \geq 1$ we see that C_{nd} is the global maximum of f . $\#$